Journal of Engineering Physics and Thermophysics, Vol. 73, No. 1, 2000

ANALYTICAL METHODS OF INVESTIGATION OF THE THERMAL STATE OF A REGION WITH A MOVING BOUNDARY UNDER THE CONDITIONS OF NONSTATIONARY HEAT TRANSFER TO THE EXTERNAL MEDIUM

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UDC 536.2

An analytical method of solution of combined problems of nonstationary heat conduction for a region with a boundary moving according to a known law and with a time-variable coefficient of heat transfer is developed. The idea of splitting the kernel of the obtained generalization of a singular integral Fourier transform with respect to a space variable provides a basis for the method. Theoretical results are used in mathematical simulation of heat transfer processes in the region with a moving boundary under the conditions of nonstationary heat transfer to the external medium.

The combined problem of nonstationary heat conduction for a region with a boundary moving according to a known law occupies a special place in mathematical theory of heat conduction [1]. The necessity of allowing for the mobility of the boundary arises, in particular, in mathematical simulation of high-temperature modes of the effect which are accompanied, for example, by destruction or ablation melting of surface layers of material [2]. Physical realization of the mentioned models of thermal effect inevitably leads to changes in the conditions of heat transfer to the external medium and manifests itself in a time variation in the coefficient of heat transfer.

The mathematical model of the heat transfer process in the region with a boundary uniformly moving according to the law $\nu \equiv \nu = 2\beta \text{Fo}$ has the form

$$\frac{\partial \theta}{\partial \text{Fo}} = \frac{\partial^2 \theta}{\partial \xi^2}, \quad \xi > \nu(\text{Fo}), \quad \text{Fo} > 0,$$
$$\theta(\xi, 0) = 0,$$
$$\left.\frac{\partial \theta(\xi, \text{Fo})}{\partial \xi}\right|_{\xi=\nu(\text{Fo})} = \text{Bi}(\text{Fo}) \left\{\theta(\xi, \text{Fo})|_{\xi=\nu(\text{Fo})} - \zeta(\text{Fo})\right\}, \tag{1}$$

where

$$\xi = \frac{x}{x_*}; \quad \text{Fo} = \frac{\kappa t}{x_*^2}; \quad \theta = \frac{T - T_0}{T_{c0} - T_0}; \quad \zeta = \frac{T_c - T_0}{T_{c0} - T_0}; \quad \text{Bi} = \frac{\alpha}{\lambda} x_* .$$

In accordance with the meaning of the problem solved, the functions Bi(Fo), ζ(Fo), and ν(Fo) are nonnegative, and the functions Bi(Fo) and ζ(Fo) are absolutely integrable on the half-open interval $[0, +\infty)$ when $\text{Fo} \geq 0$.

We emphasize that determination of an analytical solution of problem (1) involves fundamental difficulties. This is basically caused by the dependence of the function Bi on the Fourier number Fo. This problem

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Scientific-Research Institute of Special Mechanical Engineering at the N. É. Bauman Moscow State Technical University, Moscow, Russia. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 73, No. 1, pp. 125-130, January–February, 2000. Original article submitted October 23, 1998.

 1062-0125/2000/7301-0124$25.00

is discussed in [1]. It should, however, be noted that in the case of absolutely integrable functions Bi(Fo) and ζ(Fo) when $\mathrm{Fo} \geq 0$, the conditions of the theorem [3] of existence and uniqueness of the problem considered are met, i.e., the solution $\theta(\xi, \mathrm{Fo}) \in L^2[\nu(\mathrm{Fo}), +\infty)\, L^1[0, +\infty)$ of problem (1) exists and is uniqued. The present study is aimed at obtaining this solution.

To simplify further consideration, we pass over to a moving system of coordinates using a new space viariable

$$X = \xi - \nu\,(\mathrm{Fo})\,.$$

In this case, problem (1) acquires the following form:

$$\frac{\partial\theta}{\partial \mathrm{Fo}} = \frac{\partial^2\theta}{\partial X^2} + 2\beta\frac{\partial\theta}{\partial X}\,, \quad X > 0\,, \quad \mathrm{Fo} > 0\,, \tag{2}$$

$$\theta\,(X, 0) = 0\,,$$

$$\left.\frac{\partial\theta\,(X, \mathrm{Fo})}{\partial X}\right|_{X=0} = \mathrm{Bi}\,(\mathrm{Fo})\left\{\theta\,(X, \mathrm{Fo})\,|_{X=0} - \zeta\,(\mathrm{Fo})\right\}.$$

The solution of problem (2) is based on the use of a singular integral transform with respect to a space variable X:

$$u\,(\lambda, \mathrm{Fo}) \overset{\Delta}{=} \Phi\,[\theta\,(X, \mathrm{Fo})] \equiv \int_0^\infty \theta\,(X, \mathrm{Fo})\,\rho\,(X, \mathrm{Fo})\,K\,(X, \lambda, \mathrm{Fo})\,dX \equiv$$

$$\equiv \int_0^\infty \theta\,(X, \mathrm{Fo})\exp(\beta X)\left\{\cos(\lambda X) + \frac{h\,(\mathrm{Fo})}{\lambda}\sin(\lambda X)\right\}dX\,, \tag{3}$$

$$\theta\,(X, \mathrm{Fo}) = \Phi^{-1}\,[u\,(\lambda, \mathrm{Fo})] \equiv \frac{2}{\pi}\int_0^\infty u\,(\lambda, \mathrm{Fo})\exp(-\beta X)\,\times$$

$$\times\left\{\cos(\lambda X) + \frac{h\,(\mathrm{Fo})}{\lambda}\sin(\lambda X)\right\}\frac{\lambda^2 d\lambda}{\lambda^2 + h^2\,(\mathrm{Fo})}\,;$$

$$h\,(\mathrm{Fo}) \overset{\Delta}{=} \mathrm{Bi}\,(\mathrm{Fo}) + \beta\,. \tag{4}$$

Expression (3) is a generalization of the combined integral Fourier transform [4, 5].

Direct use of (3) for obtaining the solution of problem (2) is impossible [6] because its kernel $K(X, \lambda, \mathrm{Fo})$ depends not only on the space variable X and the parameter of integral transformation λ, but also on the Fourier number Fo. This, in particular, leads to the fact that

$$\Phi\left[\frac{\partial\theta\,(X, \mathrm{Fo})}{\partial \mathrm{Fo}}\right] \neq \frac{\partial u\,(\lambda, \mathrm{Fo})}{\partial \mathrm{Fo}}\,.$$

To overcome these difficulties, we use the Euler formulas and transform the kernel of integral transform (3):

$$\rho (X, \mathrm{Fo}) K (X, \lambda, \mathrm{Fo}) \equiv$$

$$\equiv \frac{1}{2} \exp (\beta X) \{\omega (\lambda, \mathrm{Fo}) \exp (i\lambda X) + \bar{\omega} (\lambda, \mathrm{Fo}) \exp (- i\lambda X)\} ,$$

where

$$\omega (\lambda, \mathrm{Fo}) \overset{\Delta}{=} 1 - i \frac{h (\mathrm{Fo})}{\lambda} ; \tag{5}$$

$$\bar{\omega} (\lambda, \mathrm{Fo}) \equiv \omega (- \lambda, \mathrm{Fo}) = 1 + i \frac{h (\mathrm{Fo})}{\lambda} .$$

Introducing the notation

$$A (\lambda, \mathrm{Fo}) \overset{\Delta}{=} \int_0^\infty \theta (X, \mathrm{Fo}) \exp [(\beta + i\lambda) X] \, dX ,$$

$$\bar{A} (\lambda, \mathrm{Fo}) \overset{\Delta}{=} \int_0^\infty \theta (X, \mathrm{Fo}) \exp [(\beta - i\lambda) X] \, dX , \tag{6}$$

we can represent the integral transform in the form

$$u (\lambda, \mathrm{Fo}) \equiv \frac{1}{2} \{\omega (\lambda, \mathrm{Fo}) A (\lambda, \mathrm{Fo}) + \bar{\omega} (\lambda, \mathrm{Fo}) \bar{A} (\lambda, \mathrm{Fo})\} \equiv$$

$$\equiv \mathrm{Re} \{\omega (\lambda, \mathrm{Fo}) A (\lambda, \mathrm{Fo})\} . \tag{7}$$

Here, according to (6), the identities

$$\bar{A} (\lambda, \mathrm{Fo}) \equiv A (- \lambda, \mathrm{Fo}) , \tag{8}$$

$$\Phi \left[\frac{\partial \theta (X, \mathrm{Fo})}{\partial \mathrm{Fo}} \right] \equiv \frac{1}{2} \left\{ \omega (\lambda, \mathrm{Fo}) \frac{\partial A (\lambda, \mathrm{Fo})}{\partial \mathrm{Fo}} + \bar{\omega} (\lambda, \mathrm{Fo}) \frac{\partial \bar{A} (\lambda, \mathrm{Fo})}{\partial \mathrm{Fo}} \right\} ,$$

$$\Phi \left[\frac{\partial^2 \theta (X, \mathrm{Fo})}{\partial X^2} + 2\beta \frac{\partial \theta (X, \mathrm{Fo})}{\partial X} \right] \equiv \mathrm{Bi} (\mathrm{Fo}) \zeta (\mathrm{Fo}) - (\lambda^2 + \beta^2) u (\lambda, \mathrm{Fo}) \equiv$$

$$\equiv \mathrm{Bi} (\mathrm{Fo}) \zeta (\mathrm{Fo}) - \frac{1}{2} (\lambda^2 + \beta^2) \{\omega (\lambda, \mathrm{Fo}) A (\lambda, \mathrm{Fo}) + \bar{\omega} (\lambda, \mathrm{Fo}) \bar{A} (\lambda, \mathrm{Fo})\}$$

hold true and problem (2) in the transforms of (3) can be put into the form

$$\omega \frac{\partial A}{\partial \mathrm{Fo}} + \bar{\omega} \frac{\partial \bar{A}}{\partial \mathrm{Fo}} + (\lambda^2 + \beta^2) \{\omega A + \bar{\omega} \bar{A}\} = 2\mathrm{Bi} (\mathrm{Fo}) \zeta (\mathrm{Fo}) , \quad \mathrm{Fo} > 0 ;$$

$$A (\lambda, 0) = 0 = \bar{A} (\lambda, 0) .$$

With regard to conditions (5) and (8) this allows one to write the Cauchy problem for the function $\mathrm{Re}\{A(\lambda, \mathrm{Fo})\}$

$$\mathrm{Re}\left\{\omega(\lambda, \mathrm{Fo})\left[\frac{\partial A(\lambda, \mathrm{Fo})}{\partial \mathrm{Fo}} + (\lambda^2+\beta^2) A(\lambda, \mathrm{Fo})\right] - \mathrm{Bi}(\mathrm{Fo})\, \zeta(\mathrm{Fo})\right\} = 0, \quad \mathrm{Fo} > 0;$$

$$A(\lambda, 0) = 0. \tag{9}$$

To determine the function $A(\lambda, \mathrm{Fo})$, we impose an additional limitation

$$\mathrm{Im}\left\{\omega(\lambda, \mathrm{Fo})\left[\frac{\partial A(\lambda, \mathrm{Fo})}{\partial \mathrm{Fo}} + (\lambda^2+\beta^2) A(\lambda, \mathrm{Fo})\right] - \mathrm{Bi}(\mathrm{Fo})\, \zeta(\mathrm{Fo})\right\} = 0, \quad \mathrm{Fo} > 0,$$

i.e., we will assume that the function $A(\lambda, \mathrm{Fo})$ is the solution of the Cauchy problem

$$\frac{\partial A(\lambda, \mathrm{Fo})}{\partial \mathrm{Fo}} + (\lambda^2+\beta^2) A(\lambda, \mathrm{Fo}) = \frac{\mathrm{Bi}(\mathrm{Fo})\, \zeta(\mathrm{Fo})}{\omega(\lambda, \mathrm{Fo})}, \quad \mathrm{Fo} > 0;$$

$$A(\lambda, 0) = 0, \tag{10}$$

where the function $\omega(\lambda, \mathrm{Fo})$ is determined by equality (5). Here it should be noted that if the function $A(\lambda, \mathrm{Fo})$ is the solution of problem (10), it is also the unknown solution of problem (9).

The solution of the Cauchy problem (10) is found by standard methods [7]:

$$A(\lambda, \mathrm{Fo}) = \int_0^{\mathrm{Fo}} \frac{\overline{\omega}(\lambda, \tau)}{|\omega(\lambda, \tau)|^2} \mathrm{Bi}(\tau)\, \zeta(\tau) \exp\left\{-(\lambda^2+\beta^2)(\mathrm{Fo}-\tau)\right\} d\tau.$$

Hence, with account for equalities (4), (5), and (7), we obtain the solution of problem (2) in the transforms of (3)

$$u(\lambda, \mathrm{Fo}) \equiv \mathrm{Re}\left\{\omega(\lambda, \mathrm{Fo})\, A(\lambda, \mathrm{Fo})\right\} =$$

$$= \int_0^{\mathrm{Fo}} \frac{\lambda^2 + h(\mathrm{Fo})\, h(\tau)}{\lambda^2 + h^2(\tau)} \mathrm{Bi}(\tau)\, \zeta(\tau) \exp\left\{-(\lambda^2+\beta^2)(\mathrm{Fo}-\tau)\right\} d\tau. \tag{11}$$

An analytically closed form of representation of the function $\theta(X, \mathrm{Fo})$ follows from (11), if we use the formula of conversion of integral transform (3)

$$\theta(X, \mathrm{Fo}) = \frac{2}{\pi} \exp(-\beta X) \int_0^{\infty} u(\lambda, \mathrm{Fo}) \left\{\cos(\lambda X) + \frac{h(\mathrm{Fo})}{\lambda} \sin(\lambda X)\right\} \times$$

$$\times \frac{\lambda^2 d\lambda}{\lambda^2 + h^2(\mathrm{Fo})}, \quad X \ge 0, \quad \mathrm{Fo} \ge 0, \tag{12}$$

where the function $u(\lambda, \mathrm{Fo})$ is determined by equality (11) and the function $h(\mathrm{Fo})$ by equality (4).

Since in the class of functions $L^2[\nu(\mathrm{Fo}), +\infty)\ L^1[0, +\infty)$ the solution of problem (2) exists and is unique, the function $\theta(X, \mathrm{Fo})$ determines a temperature field in the region $X \ge 0$ with the given law of motion of its boundary $\nu(\mathrm{Fo}) = 2\beta\mathrm{Fo}$ and the time-variable Biot number $\mathrm{Bi}(\mathrm{Fo})$ and temperature of the external medium $\zeta(\mathrm{Fo})$.

The temperature of the moving boundary can be found from (12) at $X = 0$:

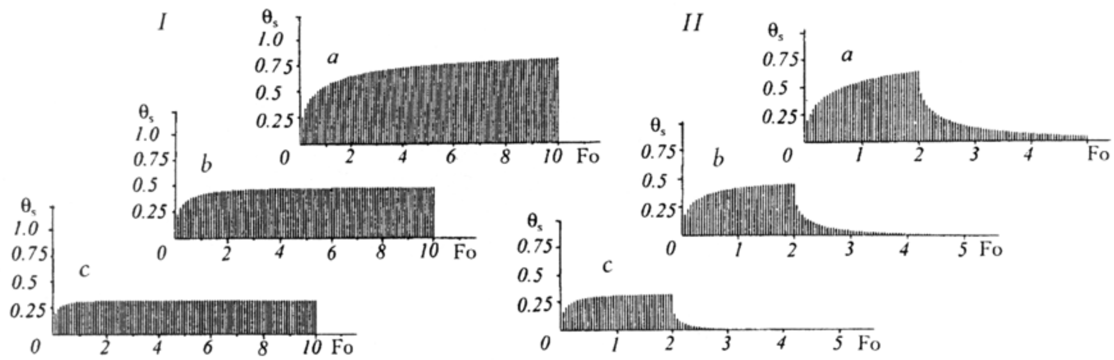

Fig. 1. Temperature θ_s of the boundary $X = 0$ of a half-space vs. Fourier number Fo under the conditions of heat transfer according to the Newton law (I) and in the pulse mode of heat transfer with duration $\tau_0 = 2$ of its "active" phase (II): a) $\beta = 0$; b) 0.5; c) 1.0 (Bi = 1).

$$\theta_s(\mathrm{Fo}) \equiv \theta(0, \mathrm{Fo}) = \frac{2}{\pi} \int_0^\infty u(\lambda, \mathrm{Fo}) \frac{\lambda^2 d\lambda}{\lambda^2 + h^2(\mathrm{Fo})}, \quad \mathrm{Fo} \ge 0 . \tag{13}$$

We use the obtained results for studying the thermal state of the region with a uniformly moving boundary in various modes of heat transfer to the external medium at $\zeta(\mathrm{Fo}) = 1$:

under the conditions of heat transfer according to the Newton law

$$\mathrm{Bi}(\mathrm{Fo}) \equiv \mathrm{Bi} = \mathrm{const}$$

and a pulse mode with the duration τ_0 of an "active" phase of heat transfer

$$\mathrm{Bi}(\mathrm{Fo}) = \mathrm{Bi}\left\{J(\mathrm{Fo}) - J(\mathrm{Fo} - \tau_0)\right\},$$

where $J(\cdot)$ is the Heaviside function [8].

In the mentioned modes of heat transfer the function $\theta_s(\mathrm{Fo})$, determined by equalities (11) and (13), can be presented in analytically explicit form:

under the conditions of heat transfer according to the Newton law

$$\theta_s(\mathrm{Fo}) = \frac{1}{\mathrm{Bi} + 2\beta} \varphi(\mathrm{Bi}, \beta, \mathrm{Fo}) J(\mathrm{Fo}), \tag{14}$$

where

$$\varphi(\mathrm{Bi}, \beta, \mathrm{Fo}) = \mathrm{Bi} + \beta \,\mathrm{erfc}\left\{\beta \sqrt{\mathrm{Fo}}\right\} - (\mathrm{Bi} + \beta) \exp\left[\mathrm{Bi}(\mathrm{Bi} + 2\beta)\mathrm{Fo}\right] \times \mathrm{erfc}\left\{(\mathrm{Bi} + \beta)\sqrt{\mathrm{Fo}}\right\};$$

in the pulse mode of heat transfer

$$\theta_s(\mathrm{Fo}) = \frac{1}{\mathrm{Bi} + 2\beta}\left\{\varphi(\mathrm{Bi}, \beta, \mathrm{Fo}) J(\mathrm{Fo}) - \varphi(\mathrm{Bi}, \beta, \mathrm{Fo} - \tau_0) J(\mathrm{Fo} - \tau_0)\right\},$$

where $\mathrm{erfc}\, u = \frac{2}{\sqrt{\pi}} \int_u^\infty \exp(-z^2) dz$ is the complementary Gauss error function [8].

At $\beta = 0$ ($\dot{v} = 0$), the solutions obtained determine the temperature of a motionless boundary $X \equiv \xi = 0$ of a half-space in the studied modes of heat transfer to the external medium.

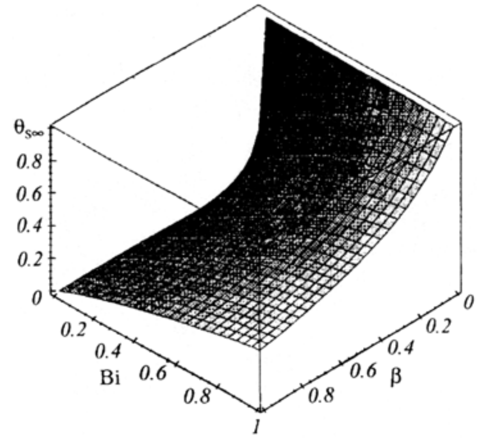

Fig. 2. Temperature $\theta_{s\infty}$ vs. parameter β and Biot number Bi under the conditions of heat transfer according to the Newton law.

Figure 1 presents time dependences of the temperature of a moving boundary $X = 0$ under the conditions of heat transfer according to the Newton law (I) and in the pulse mode of heat transfer $\tau_0 = 2$ (II) at Bi = 1 and different values of the parameter β. It is seen that the mobility of the boundary of the region reduces the magnitude of its heating.

At higher values of time the asymptotic behavior of the function $\theta_s(\text{Fo})$ assigned by equality (14) is determined as

$$\theta_s(\text{Fo}) \sim \frac{\text{Bi}}{\text{Bi} + 2\beta}\left\{1 - \frac{\text{Bi}(\text{Bi} + 2\beta)}{2\sqrt{\pi}\,\beta^2(\text{Bi} + \beta)^2\,\text{Fo}^{3/2}}\exp(-\beta^2\,\text{Fo})\right\},$$

i.e., when $\text{Fo} \to +\infty$

$$\theta_{s\infty} = \frac{\text{Bi}}{\text{Bi} + 2\beta}.$$

Hence it follows that under the conditions of heat transfer according to the Newton law, the maximum heating of the region with a moving boundary is determined by the velocity of boundary motion and depends on the Biot number (Fig. 2).

We note for comparison that the asymptotic estimate of the maximum heating of the region with a motionless boundary ($\beta = 0$) under the conditions of heat transfer according to the Newton law has the form [8]

$$\theta_s(\text{Fo}) \sim 1 - \frac{1}{\text{Bi}\sqrt{\pi \text{Fo}}},$$

i.e., $\theta_{s\infty} = 1$ when $\text{Fo} \to \infty$.

In the pulse mode of heat transfer to the external medium $\theta_s(\text{Fo}) \to 0$, when $\text{Fo} \to +\infty$ for all $\beta \geq 0$.

This work was carried out with financial support from the Russian Fund for Fundamental Research, project code 96-03-32193.

NOTATION

x, space variable; t, time; T, temperature of half-space; T_e, temperature of external medium; $\xi = x/x_*$, dimensionless variable; $\text{Fo} = \kappa t/x_*^2$, Fourier number; $\text{Bi} = \alpha x_*/\lambda$, Biot number; x_*, selected unit of scale; λ, thermal conductivity; κ, thermal diffusivity; α, coefficient of heat transfer.

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